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SPECTRAL REPRESENTATION OF GENERALIZED TRANSITION KERNELS

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ABSTRACT

Let $(X, \mathcal{A})$ be a set with a countably σ-generated "Borel" field of subsets; let W be a "Borel" subset of the product of $(X, \mathcal{A})$ with the real line $\mathbb{R}$ and its Borel field $\mathcal{B}$; and for each $x \in X$ let γ_x be a measure on the "slice" $W_x = \{(w,t) \in W: w = x\}$. It is shown that, under reasonable conditions, the σ-field $\mathcal{A} \otimes \mathcal{B}|W$ can be generated by a real-valued function g in such a way that, given any measurable $f: W \to \mathbb{R}$, g can be chosen to be arbitrarily close to f and so that its "slice-integrals" $\int_{W_x} g(x,t)d\gamma_x(t)(x \in X)$ coincide with those of f. This theorem is the first step in a study of monotonic sequences of countably generated σ-fields.

1. Introduction

Suppose T is an endomorphism of a Polish measure space $(X, \mathcal{B}, \mu)$ and (for $n = 0,1,2,\ldots$) $\mathcal{A}_n = T^{-n}(\mathcal{B})$. The original motivation for the present work was a desire to improve the usual "planar representation", of the disintegration of $\mathcal{B}$ over $\mathcal{A}_1$, to a simultaneous representation in the Hilbert cube of all the disintegrations of $\mathcal{A}_n$ over $\mathcal{A}_{n+1}$. This leads more generally to an investigation of the structure of a fairly arbitrary monotone sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \ldots$ of σ-subfields of $\mathcal{B}$ $(= \mathcal{B}_0)$. A useful tool here would be a description of the sequence in terms of a martingale (or decreasing martingale) $g_0, g_1, \ldots, g_n, \ldots$, "adapted" to the sequence; that is, each $\mathcal{B}_n$ is to be spectrally generated by the corresponding g_n. In a subsequent paper [7] the author hopes to use the results of the present paper to carry out this program for a reasonably general increasing sequence of

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σ-fields, and also (more interestingly but less completely) for a *finite* decreasing sequence. (For an *infinite* decreasing sequence, the result has been obtained only in a special, though significant, case.)

The present paper is concerned with a uniformly σ-finite sectioned "generalized transition kernel" [5] or "conditional measure distribution" [3] on a set X. In effect this consists of a subset W of $X \times \mathbb{R}$, with a suitable "product Borel field" $\mathcal{C}$ of "measurable" sets, and measures γ_x ($x \in X$) on the respective "slices" $W_x = (\{x\} \times \mathbb{R}) \cap W$, subject to appropriate measurability conditions. The result, roughly, is that an arbitrary measurable function f: $X \to W$ can be approximated arbitrarily closely by a one-to-one measurable function g having the same "slice-integrals" (i.e., such that $\int_{W_x} g d\gamma_x = \int_{W_x} f d\gamma_x$ for all x). A precise statement of the theorem is in Section 2 below. It constitutes a considerable generalization of the main theorem of [4], which is essentially the special case of the present theorem in which each W_x consists of two atoms. The "arbitrarily close approximation" here is in the sense of the topology of close approximation [1], which has the advantage that the theorem is invariant if the measures are replaced by equivalent ones.

In view of the application [7] the theorem is actually formulated in terms of "spectral generation" by g, rather than by requiring g (as in the rough statement above) to be a measurable injection. That requirement would automatically force the σ-field $\mathcal{C}$ to be countably separated; and in [4] this restriction on $\mathcal{C}$ was part of the hypothesis. Here we require instead that $\mathcal{C}$ is to be "spectrally generated" by g — that is, $\mathcal{C}$ consists of the sets $g^{-1}(B)$ where $B \in \mathcal{B}(\mathbb{R})$, the family of Borel subsets of the real line. Thus we must assume that $\mathcal{C}$, instead of being countably separated, is countably (σ-) generated. (Conversely, it is well known that every countably generated—by which we always mean countably σ-generated—σ-field of sets is spectrally generated by some function.) The present hypothesis of countable generation is in a sense stronger than countable separation, because one can always make a countable generating system act as a separating system too, simply by identifying points not separated. And the conclusion will be correspondingly stronger, because the generating function g will automatically be one-to-one on the resulting quotient space. In "nice" cases (e.g., the Borel sets in a Polish space) the σ-field $\mathcal{C}$ will have the Blackwell property, and the two formulations become equivalent. It should also be remarked that countable separation is usually not inherited by σ-subfields; countable generation is in practice more likely to be inherited —as, for example, by cylinder sets in a product space.

The analogous theorem for a countably separated $\mathcal{C}$ and a measurable injection g will also hold, with only minor alterations in the proofs. And the arguments in [4] apply almost unchanged to replace "one-to-one" by "spectrally generating" in the countably generated case, thereby providing an essential starting point for the present work.

The proof proceeds by applying [5] to replace W by a "planar representation", which is then partitioned into countably many simpler pieces, which are dealt with separately. This is the technique used in [4]; it requires precautions to prevent the separate spectral representations from interfering with one another. The device for doing this, justified in [4], is that the partial functions can be made to satisfy a somewhat complicated "range restriction" (see 2.2(iii) below); for short, we say that they are "range restricted". In [4] such restrictions were used to ensure that the injective partial functions fitted together to form an injective global one; but it is easy to see that they also ensure that the generating partial functions fit together to form a generating global one.

2. The theorem

2.1. The setting can be described as follows (cf. [5, §§1 and 9]). Let $\mathcal{A}$ be a σ-field of subsets of a set X, and form the product $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B})$, where $\mathcal{B} = \mathcal{B}(\mathbb{R})$ and $\mathcal{A} \otimes \mathcal{B}$ is the "product σ-field" (σ-) generated by the sets $A \times B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$. We refer to the members of $\mathcal{A} \otimes \mathcal{B}$ as the "Borel" or "measurable" sets. Fix a set $W \in \mathcal{A} \otimes \mathcal{B}$. The (generalized) transition kernel γ is a map assigning to each $x \in X$ a (non-negative, countably additive, σ-finite) measure γ_x on $\mathcal{A} \otimes \mathcal{B}$, with $\gamma_x(L)$ depending only on the intersection (possibly empty) of L with the "slice" $W_x = \{(x,t) \in W: t \in \mathbb{R}\}$ (also possibly empty), in such a way that, for each $L \in \mathcal{A} \otimes \mathcal{B}$, the function $x \mapsto \gamma_x(L)$ is $\mathcal{A}$-measurable. It is often convenient to think of γ_x as defined on the Borel sets of the x-slice W_x ($x \in X$); thus we may write $\gamma_x(L)$ as $\gamma_x(L_x)$, where L_x is the x-slice of $L \in \mathcal{A} \otimes \mathcal{B}$. (Caution: The "slices" L_x and W_x need not belong to $\mathcal{A} \otimes \mathcal{B}$, because we do not assume that $\mathcal{A}$ contains all singletons. However, the sets $\{t \in \mathbb{R}: (x,t) \in L\}$ will be Borel subsets of $\mathbb{R}$.)

The emphasis in [5] was on the case in which $W = X \times [0,1]$ and each γ_x is finite; but the greater generality (as in [5, §§9]) will be convenient for applications [7]. On the other hand, we shall specialize matters in three respects:

(1) We shall assume throughout that $\mathcal{A}$ (which in [5] was an arbitrary σ-field) is countably (σ-) generated.

(2) We assume that γ is *uniformly σ-finite* on W; that is, W is expressible as $\bigcup\{L^n\colon n \in \mathbb{N}\}$, where each L^n is "Borel" and $\gamma_x(L^n) < \infty$ for all $x \in X$ and $n \in \mathbb{N}$. (This is substantially equivalent to the definition in [3, p. 148], in view of the remarks in [5, p. 282]. The notations differ somewhat confusingly; the "X" in [3] corresponds to "$X \times \mathbb{R}$" in [5], and "Y" in [3] is "X" in [5]. We follow [5] here.)

(3) Define a "Dirac measure" to be one assigning a positive value to some singleton and zero to its complement. We assume no γ_x is Dirac.

2.2. THEOREM 1: *Assume the set-up in* 2.1 *(and in particular the assumptions* (1), (2), (3)*). Suppose given*

(a) *two $\mathcal{A} \otimes \mathcal{B}$-measurable real-valued functions f, ε, on W, of which ε is everywhere positive, and*

(b) *a G_δ subset H of $\mathbb{R}$ containing $\mathbb{Q}$ (the rationals).*

Then there is an $\mathcal{A} \otimes \mathcal{B}$-measurable real-valued function g on W such that

(i) *the spectral field of g is $\mathcal{A} \otimes \mathcal{B}|W$, the family of all "Borel" subsets of W,*

(ii) $|f(x,t) - g(x,t)| < \varepsilon(x,t)$ *for all* $(x,t) \in W$,

(iii) *(the "range restriction") for some F_σ set M we have $g(W) \subset M \subset H \setminus \mathbb{Q}$, and*

(iv) *for each $x \in X$, $\int_{W_X} f(x,t)d\gamma_x(t) = \int_{W_x} g(x,t)d\gamma_x(t)$ whenever either side is defined.*

(Here an integral is "defined" by integrating the positive part and the negative part separately and taking the difference provided this does not involve $\infty - \infty$.)

Remarks: (a) As mentioned in Section 1, the assumption 2.1(1), that $\mathcal{A}$ is countably generated, could be replaced by the assumption that $\mathcal{A}$ is countably separated, with (i) then being replaced by

(i′) g is injective.

The proof would be essentially the same.

(b) The complications in the "range restriction" (iii), involving $\mathbb{Q}$, are required (and justified) by the construction in [4].

(c) The assumption (1) and (3), in 2.1, are not superfluous.

In fact, (1) is trivially needed if (i) above is to hold. For (3), it is easy to see that *two* Dirac measures $\gamma_{x_0}, \gamma_{x_1}$ (with $x_0 \neq x_1$) would also make (i) above impossible in general. However, *one* Dirac measure could be allowed, at the cost of complicating the argument.

It would be interesting to know whether assumption (2) is needed. (It is superfluous if each γ_x is locally finite, by [2].) Its main significance here is that it provides a satisfactory distribution function for γ (cf. [5]).

(d) It clearly suffices to prove Theorem 1 with ε replaced by something smaller. Thus we shall assume, for convenience, throughout what follows, that

$$\int_{W_x} \varepsilon(x,t)d\gamma_x(t) < \infty \quad (x \in X) \tag{1}$$

and

$$\begin{aligned}&\varepsilon(x,t) = 1/k(x,t), \text{where } k \text{ is a measurable}\\ &\text{function taking positive integer values only.}\end{aligned} \tag{2}$$

2.3. The strategy of the proof of Theorem 1 is to decompose W into countably many pieces for each of which the conclusion of the theorem holds, and to deduce that it holds for W. More precisely, we prove:

LEMMA: *In the notation of* 2.1 *and* 2.2, *suppose* $W = \bigcup\{W_n\colon n \in \mathbb{N}\}$ *where the sets* W_n *are "Borel" and pairwise disjoint and satisfy*

(a) $\gamma_x(W_n) < \infty$ *for all* $x \in X$ *and* $n \in \mathbb{N}$,
(b) *for each* $x \in X$, $f|(W_n)_x$ *is bounded,*
(c) *for each* n *and each* G_δ *set* $H_n \supset \mathbb{Q}$, *there is a function* $g_n\colon W_n \to \mathbb{R}$ *such that*
  (i) *the spectral field of* g_n *is* $(\mathcal{A} \otimes \mathcal{B})|W_n$,
  (ii) $|f(x,t) - g_n(x,t)| < \varepsilon(x,t)$ *for all* $(x,t) \in W_n$,
  (iii) $g_n(W_n) \subset$ *some* F_σ *subset* M_n *of* $H_n \setminus \mathbb{Q}$,
  (iv) *for each* $x \in X$, $\int_{(W_n)_x} f(x,t)d\gamma_x(t) = \int_{(W_n)_x} g_n(x,t)d\gamma_x(t)$.

Then Theorem 1 holds (for W*).*

Proof: Define specific sets H_n recursively, as follows. Take H_1 to be the set H of Theorem 1 (2.2(b)). From condition (c) above we obtain $g_1\colon W_1 \to \mathbb{R}$ with spectral field $(\mathcal{A} \otimes \mathcal{B})|W_1$, satisfying conditions (i)–(iv) for $n = 1$. When $g_n\colon W_n \to \mathbb{R}$ has been defined, with properties (i)–(iv), put $H_{n+1} = H_n \setminus M_n$ and use (c) to obtain g_{n+1}. This defines g_n for all $n \in \mathbb{N}$, and we put $g = \bigcup_n g_n$ (that is, $g(x,t) = g_n(x,t)$ when $(x,t) \in W_n$). It is a straightforward matter to verify that this works. ∎

Remark: In the Lemma there is no need to exclude Dirac measures from the measures $\gamma_x|W_n$. They will have to be excluded, however, when the sets W_n are constructed (cf. 3.2 and 3.8 below).

3. The "planar" representation

3.1. As shown in [5, 9.5], we can arrange (by applying a first-coordinate-preserving "Borel" isomorphism) that W and γ have the following special form. W now consists of the union of three pairwise disjoint $\mathcal{A} \otimes \mathcal{B}$-measurable sets, $S \cup T \cup \mathcal{O}'$, such that

(1) the "singular set" S is a subset of $X \times K^0$, where K^0 is a (Lebesgue) null Cantor set contained in (-1, 0),

(2) the "atomic set" T is the union of sets $A_n \times \{-n\}$, where $n = 1, 2, \ldots$, and $A_n \in \mathcal{A}$, and

(3) the "almost ordinate set" $\mathcal{O}'$ is of the form $\mathcal{O} \setminus N$ where $\mathcal{O} = \{(x,t)\colon x \in X^c, 0 \le t < \Phi(x)\}$ for some $X^c \in \mathcal{A}$ and some $\mathcal{A}$-measurable $\Phi\colon X^c \to (0, \infty]$ and N is a measurable ("Borel") subset of $\mathcal{O}$.

The measure γ_x (thought of as located on the x-slice W_x) is zero on S_x, purely atomic on T_x (each singleton $(x, -n)$, $x \in A_n$, having positive γ_x) and is linear Lebesgue measure λ on each $\mathcal{O}'_x$; further, $\lambda(N_x) = 0$ for each x (N is "fully null").

We further rearrange T so that

(4) $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$.

This property was achieved in [5, 8.1] for the case in which all the measures γ_x are finite. It was lost in the extension to the uniformly σ-finite case, but can be restored as follows. Write

$$B_1 = \bigcup_n A_n = A_1 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_n \setminus (A_1 \cup \cdots \cup A_{n-1})) \cup \cdots$$

and

$$T_1 = (A_1 \times \{-1\}) \cup ((A_2 \setminus A_1) \times \{-2\}) \cup \cdots \cup ((A_n \setminus (A_1 \cup \cdots \cup A_{n-1})) \times \{-n\} \cup \cdots .$$

Map T_1 onto $B_1 \times \{-1\}$ by vertical projection. Now $T \setminus T_1$ is of the form $\bigcup \{A'_n \times \{-n\}\colon n = 2, 3, \ldots\}$, where $A'_n \subset A_n$. Repeat the process, mapping the subset

$$T_2 = (A'_2 \times \{-2\}) \cup ((A'_3 \setminus A'_2) \times \{-3\}) \cup \cdots \cup ((A'_n \setminus (A'_2 \cup \cdots \cup A'_{n-1})) \times \{-n\}) \cup \cdots$$

of $T \setminus T_1$ by vertical projection onto $B_2 \times \{-2\}$, where

$$B_2 = \bigcup\{A'_n : n \geq 2\} \subset B_1;$$

and so on. This produces a first-coordinate-preserving isomorphism of T onto $\bigcup\{B_n \times \{-n\}: n = 1, 2, \dots\}$ where $B_1 \supset B_2 \supset \cdots$; and we replace the sets A_n by the B_n's. (One can verify that B_n = union of all intersections of n of the sets $A_1, A_2, \dots$.)

3.2. Define $C_n = A_n \setminus A_{n+1}, C_\infty = \bigcap\{A_n : n \in \mathbb{N}\}$; thus C_n is the set of x's for which γ_x has exactly n atoms, namely atoms at the points $(x, -i)$, $1 \leq i \leq n$, and analogously for C_∞. Write $C_n^* = C_n \times \{-1, -2, \dots, -n\}$, and $C_\infty^* = C_\infty \times (-\mathbb{N})$, forming a partition of T. Since there are no Dirac measures γ_x, C_1 must be a subset of the projection $\pi(\mathcal{O}') = X^c$. We shall combine C_1^* with a subset of $\mathcal{O}'$ and deal with it later (3.8). The rest of T, namely

$$T \setminus C_1^* = C_\infty^* \cup \bigcup\{C_n^* : n \geq 2\},$$

will now be partitioned into pieces to which Lemma 2.3 will apply, by means of the following lemma.

LEMMA: *Given $C \in \mathcal{A}$, and $2r + 2$ $\mathcal{A}$-measurable real-valued functions $f, h_1, h_2, \dots, h_r$, $c_1, c_2, \dots, c_r, \varepsilon$ on C, where $r \geq 2$, such that $c_1, c_2, \dots, c_r$ and ε are everywhere positive and $f = h_1 + h_2 + \cdots + h_r$, then there exist range-restricted functions $u_1, u_2, \dots, u_r$ on C, with pairwise disjoint ranges, such that*

(i) $f = c_1u_1 + c_2u_2 + \cdots + c_ru_r$,

(ii) $|c_i(x)u_i(x) - h_i(x)| < \varepsilon(x)$ *for all* $x \in C$ *and* $i = 1, 2, \dots, r$, *and*

(iii) *each* u_i *spectrally generates the* σ*-field* $\mathcal{A}|C$.

Remark: As stated in Section 1 (cf. also [4]), the term "range-restricted" here serves as an abbreviation for: "Given a G_δ set H_i containing $\mathbb{Q}$, the range $u_i(C)$ is to be contained in some F_σ subset M_i of $H_i \setminus \mathbb{Q}$".

Proof of Lemma: For $r = 2$ the assertion of the Lemma is essentially the Main Theorem (Case A) of [4]; the general case follows by a routine induction argument.

Incidentally, an analogous Lemma is also true for $\aleph_0$ functions h_i, but it does not help in the present context, since the set C_∞^* has to be partitioned further because of the finiteness requirements in Lemma 2.3.

3.3. On the subset C_r^* of T, where $r \geq 2$, Lemma 3.2 shows that the hypotheses of Lemma 2.3 are fulfilled (conditions 2.3 (a) and (b) hold because each $(C_r^*)_x$ is finite). We have f and ε given, and take $h_1, \ldots, h_r$ in 3.2 to be all equal to f/r. We take $c_i = \gamma_i(x, -i)$ and define the required spectrally-generating range-restricted function by $g(x, -i) = u_i(x) \quad (i = 1, \ldots, r)$.

Finally we partition C_∞^* into the sets

$$C_{\infty s}^* = C_\infty \times \{-(2s-1), -2s\}, \quad s = 1, 2, \ldots ,$$

and apply the argument for $r = 2$ to each $C_{\infty s}^*$. Of course, Lemma 2.3 applies to these sets too.

3.4. The "singular set" S is easily dealt with by means of the following lemma (which will also be of use later).

LEMMA: *Let Y be an $\mathcal{A} \otimes \mathcal{B}$-measurable subset of $A \times \mathbb{R}$, where $A \in \mathcal{A}$, such that $\lambda(Y_x) = 0$ for all $x \in A$. Given measurable functions f: $Y \to \mathbb{R}$ and ε: $Y \to (0, \infty)$, there is a range-restricted measurable function g: $Y \to \mathbb{R}$, whose spectral field is $\mathcal{A} \times \mathcal{B}|Y$, such that $|f(x,t) - g(x,t)| < \varepsilon(x,t)$ for all $(x,t) \in Y$ and, for each $x \in A$,*

$$\int_{Y_x} f(x,t)d\lambda(t) = \int_{Y_x} g(x,t)d\lambda(t).$$

Proof: From [4. Cor. 2] we obtain a range-restricted g: $Y \to \mathbb{R}$ that is ε-close to f, and (as usual) we can replace the injectivity of g in [4] by spectral generation. The final requirement holds automatically and trivially, for here both integrals are 0. ∎

3.5. As mentioned in 3.2, the subset $W \cap (C_1 \times \mathbb{R})$ of W will need special attention (to avoid introducing Dirac measures), and we postpone its consideration until 3.8. To partition the rest of $\mathcal{O}'$ appropriately, we shall rely on the following lemma.

LEMMA: *Let Y be an $\mathcal{A} \times \mathcal{B}$-measurable subset of $A \times \mathbb{R}$, where $A \in \mathcal{A}$, such that $0 < \lambda(Y_x) < \infty$ for all $x \in A$. Given measurable functions f: $Y \to \mathbb{R}$, α: $A \to \mathbb{R}$ and a constant $\eta \in (0,1)$ such that $\alpha(x) \leq f(x,t) \leq \alpha(x) + \eta$ for all $(x,t) \in Y$, there exists a range-restricted measurable g: $Y \to \mathbb{R}$, with spectral field $\mathcal{A} \otimes \mathcal{B}|Y$, such that $|f(x,t) - g(x,t)| < 2\eta$ for all $(x,t) \in Y$ and (for each $x \in A$)*

$$\int_{Y_x} f(x,t)d\lambda(t) = \int_{Y_x} g(x,t)d\lambda(t).$$

We first prove this in a special case, in which Y is a "lower ordinate set" of the form $\{(x,t)\colon x \in A,\ 0 \le t < \phi(t)\}$ for some measurable $\phi\colon A \to (0,\infty)$. (Thus here $\phi(x) = \lambda(Y_x)$.)

Proof: (in special case) Apply [4, Theorem 1], taking "X" of that theorem to be Y, "f" to be f, "g" $= f/2 =$ "h", "c" $= 1/2 =$ "d", and "ε" $= \eta/2$. This gives range-restricted functions u, w on Y, with disjoint ranges, such that (for all $(x,t) \in Y$) $f(x,t) = \frac{1}{2}u(x) + \frac{1}{2}w(x)$ and

$$|f(x,t) - u(x,t)| = |f(x,t) - w(x,t)| < \eta.$$

Further, both u and w ("injective" in [4]) can here be taken to be spectrally generating for $\mathcal{A} \otimes \mathcal{B}|Y$.

Write

$$Y' = \{(x,t) \in Y\colon 0 \le t < \phi(x)/2\}, \quad Y'' = \{(x,t) \in Y\colon \phi(x)/2 \le t < \phi(x)\},$$

and define $g\colon Y \to \mathbb{R}$ by

$$g(x,t) = \begin{cases} u(x,2t) & \text{if } (x,t) \in Y', \\ w(x, 2t - \phi(x) & \text{if } (x,t) \in Y''. \end{cases}$$

It is easy to see that g has the required properties. ∎

3.6. Next we remove the condition that Y is a lower ordinate set, but obtain a slightly weaker conclusion, applying only to $Y \setminus N'$ where N' is "fully null" (i.e., $\lambda(N'_x) = 0$ for all x). With the original hypotheses of Lemma 3.5, define (for $x \in A$ and $t \ge 0$)

$$F(x,t) = \lambda(Y_x \cap [0,t)) = \lambda(Y_x \cap [0,t]).$$

This is $\mathcal{A}$-measurable in x for fixed t, and continuous and increasing (= non-decreasing) in t for fixed x, so it is $\mathcal{A} \otimes \mathcal{B}$-measurable [6, 3.2]. Hence so also is $\tilde{F}$, where $\tilde{F}(x,t) = (x, F(x,t))$ (see [6], loc. cit.).

The "extended constancy set" of F (the union of the closures of the intervals of constancy for each x) is

$$C'^*(F) = \bigcup\{\{x\} \times [(F_x)_{\leftarrow}(p),\ (F_x)^{\leftarrow}(p)]\colon p \in K(F_x), x \in A\}$$

(ibid. p. 10), where

$$K(F_x) = \{p \in R\colon (F_x)^{-1}(p) \text{ has more than one point}\},$$

and is likewise measurable.

Write $N^* = Y \cap C'^*(F)$, a fully null "Borel" set, and put $Y^* = Y \setminus N^*$. Let Z^+ and Z^0 be the upper and lower ordinate sets of the function $x \to \lambda(Y_x)$ $(x \in A)$; that is,

$$Z^+ = \{(x,t)\colon x \in A,\ 0 \le t \le \lambda(Y_x)\}, \quad Z^0 = \{(x,t)\colon x \in A,\ 0 \le t < \lambda(Y_x)\}.$$

Note that here Y^* is the "good set" $G^*(F)$ of [6, 3.5], since here the "jump set" $J^*(F)$ is empty (because F_x is continuous in t). As remarked in [6, p. 11] (modulo a misprint confusing $\tilde{F}$ with F), $\tilde{F}|Y^*$ is therefore a bijection, onto an $\mathcal{A}\otimes\mathcal{B}$-measurable set (namely $G^*(F^{\leftarrow})$), and is measurable and has a measurable inverse. Since $\tilde{F}|Y^*$ preserves Lebesgue measure on each x-slice, we have that $\tilde{F}(Y^*) \subset Z^+$, and $Z^+ \setminus \tilde{F}(Y^*)$ is fully null.

Put $Z^- = \tilde{F}(Y^*) \cap Z^0$. The three sets $\tilde{F}(Y^*)$, Z^+, Z^0 differ only by fully null sets, so that Z^- is an "almost ordinate set".

We abbreviate the restriction $\tilde{F}|Y^*$ to $\tilde{F}^*$, and define $f'\colon Z^0 \to \mathbb{R}$ by

$$f'(x,t) = \begin{cases} f((\tilde{F}^*)^{-1}(x,t)) & \text{if } (x,t) \in Z^-, \\ \alpha(x) & \text{if } (x,t) \in Z^0 \setminus Z^-. \end{cases}$$

From the special case of Lemma 3.5 proved above, there is a range-restricted measurable $g'\colon Z^0 \to \mathbb{R}$, generating $\mathcal{A}\otimes\mathcal{B}|Z^0$, such that $|g'(x,t) - f'(x,t)| < 2\eta$ for all $(x,t) \in Z^0$, and such that, for each $x \in A$,

$$\int_{Z^0_x} g'(x,t)d\lambda(t) = \int_{Z^0_x} f'(x,t)d\lambda(t).$$

Write $(\tilde{F}^*)^{-1}(Z^0) = Y'$; this is of the form $Y \setminus N'$ where N' is fully null. Define $g\colon Y' \to \mathbb{R}$ by

$$g(x,t) = g'(\tilde{F}(x,t)).$$

It is easy to check that g has the required properties for Y' (rather than Y).

3.7. To complete the proof of Lemma 3.5 as first stated, we merely use Lemma 3.4 to define a suitably range-restricted function on the omitted fully null set N'. This combines with the function produced by 3.6 to give a function satisfying all the requirements of Lemma 3.5 (because the "slice integrals" over each N'_x are zero).

3.8. We combine the single-atom subset C_1^* with a subset of the almost-ordinate set $\mathcal{O}'$ as follows. Recall that $\mathcal{O}' = \mathcal{O} \setminus N$, where $\mathcal{O}$ is the lower ordinate set of a positive (possibly infinite) measurable function $\Phi: A \to (0, \infty]$ and N is a fully null subset of $\mathcal{O}$. We shall define a subset P^* of $\mathcal{O}'$ and produce a suitable partitioning (to which Lemma 2.3 will apply) of $P^* \cup C_1^*$.

We use the notation $\pi_e(Y)$ for the "essential projection" $\{x \in X: \lambda(Y_x) > 0\}$ of an $\mathcal{A} \otimes \mathcal{B}$-measurable subset Y of $X \times [0, \infty)$, and $e(Y)$ for the "essential part" $Y \cap (\pi_e(Y) \times \mathbb{R})$ of Y. Note that $\pi_e(Y) \in \mathcal{A}$, because of the measurability condition satisfied by transition kernels. Thus $e(Y) \in \mathcal{A} \otimes \mathcal{B}$, and $Y \setminus e(Y)$ is fully null. We abbreviate $\gamma_x(x, -1)$ to $c(x)$ (where $x \in C$); this is a positive finite $\mathcal{A}$-measurable function. Recall (2.2(2)) that $1/\varepsilon(x,t)$ is integer-valued, and write (for $k \in \mathbb{N} = \{1, 2, \dots\}$ and $r \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$)

$$U_{kr} = \{(x,t) \in \mathcal{O}', x \in C_1, t < 1,\ \varepsilon(x,t) = 1/k,\ r \leq 4kf(x,t) < r+1\},$$

forming a countable family of pairwise disjoint measurable sets with union $\mathcal{O}' \cap (X \times [0,1))$. Enumerate the sets U_{kr} into a single sequence, say $V_1, V_2, \dots$, where $V_n = U_{k(n)r(n)}$, and consider the pairwise disjoint sets

$$Z_n = \pi_e(V_n) \setminus \bigcup \{\pi_e(V_j): j < n\}, \quad n \in \mathbb{N}.$$

We have $C_1 \subset A$ because no γ_x is a Dirac measure (2.1(3)), so if $x \in C_1$ then $\lambda(\mathcal{O}' \cap (X \times [0,1)))_x = \min(\Phi(x), 1) > 0$. Thus $\lambda(V_n)_x > 0$ for a first n, say $n(x)$, so that $x \in Z_{n(x)}$. Hence

$$\bigcup\{Z_n: n \in \mathbb{N}\} = C_1. \tag{1}$$

Put $Z_n^* = V_n \cap (Z_n \times \mathbb{R})$ $(n \in \mathbb{N})$. The sets $Z_1^*, Z_2^*, \dots$ are pairwise disjoint and $\mathcal{A} \times \mathcal{B}$-measurable. Denote their union by P_1^*; thus $P_1^* \subset \mathcal{O}'$, and $\pi_e(P^*) = C_1$ because $\pi_e(Z_n^*) = Z_n$. In particular,

$$\lambda(Z_n^*)_x > 0 \quad \text{if } x \in Z_n. \tag{2}$$

We have $Z_n^* \subset V_n = U_{kr}$ for some $k = k(n)$ and $r = r(n)$, and therefore

$$\text{for all } (x,t) \in Z_n^*, \quad \varepsilon(x,t) = 1/k \quad \text{and} \quad \frac{r}{4k} \leq f(x,t) < \frac{r+1}{4k}. \tag{3}$$

Now, for $n, h, j \in \mathbb{N}$, define

$$C_{nhj} = \{x \in Z_n : \varepsilon(x,-1) = 1/h,\ j-1 < c(x)/\lambda(Z_n^*)_x \le j\},$$
$$C_{nhj}^* = C_{nhj} \times \{-1\},$$
$$P_{nhj} = Z_n^* \cap (C_{nhj} \times \mathbb{R}), \quad Q_{nhj} = P_{nhj} \cup C_{nhj}^*.$$

The sets C_{nhj} form an $\mathcal{A}$-measurable partition of C_1, and the sets P_{nhj} form an $\mathcal{A}\otimes\mathcal{B}$-measurable partition of P^*. Thus the sets Q_{nhj} form an $\mathcal{A}\otimes\mathcal{B}$-measurable partition of $P^* \cup C_1^*$. Note that if $x \in C_{nhj}$ then $(P_{nhj})_x = (Z_n^*)_x$. Now consider a particular Q_{nhj}. The suffixes n, h, j will remain fixed until the end of the argument, so to save notation we omit them, writing simply $Q = P \cup C^*$ for $Q_{nhj} = P_{nhj} \cup C_{nhj}^*$. Accordingly we have

(4) if $(x,-1) \in C^*$ then $\varepsilon(x,-1) = 1/h$ and $j-1 < c(x)/\lambda(P_x) \le j$.

Apply [4, Cor. 2, p. 152] to the function f on C^*, taking "ε" (in the Corollary) to be $\inf(1/h, 1/2jk)$, where $k = k(n)$. This provides a range-restricted spectrally generating function g_{-1} on C $(= C_{nhj})$ such that

(5) $$|g_{-1}(x) - f(x,-1)| < \inf(1/h, 1/2jk) \quad (x \in C).$$

Write $\beta(x) = (f(x,-1) - g_{-1}(x))c(x)/\lambda(P_x)$ $(x \in C)$; from (4) and (5) we have $|\beta(x)| < 1/2k$. Define

$$\bar{f}(x,t) = f(x,t) + \beta(x) \quad \text{for all } (x,t) \in P.$$

Apply Lemma 3.5 to the function "f" $= \bar{f}$ on "Y" $= P$, taking $\alpha(x) = \beta(x) + r/4k$ and $\eta = 1/4k$, and noting that $\alpha(x) \le \bar{f}(x,t) \le \alpha(x) + \eta$ when $(x,t) \in P$, as Lemma 3.5 requires. This gives a range-restricted spectrally generating function $\bar{g}$ on P, with range disjoint from that of g_{-1}, such that $|\bar{f}(x,t) - \bar{g}(x,t)| < 2\eta = 1/2k$ and (for each $x \in C$)

$$\int_{P_x} \bar{g}(x,t)\,d\lambda(t) = \int_{P_x} \bar{f}(x,t)\,d\lambda(t).$$

Finally, define $g: Q \to \mathbb{R}$ by

$$g(x,t) = \bar{g}(x,t) \quad \text{if } (x,t) \in P,$$
$$g(x,-1) = g_{-1}(x) \quad \text{if } (x,-1) \in C^*.$$

Then g is a range-restricted generating function for $\mathcal{A}\otimes\mathcal{B}|Q$, differing from f by less than ε. And we have

$$\begin{aligned}\int_{Q_x}(g(x,t)-f(x,t))d\gamma_x(t) =& c(x)(g_{-1}(x)-f(x,-1))\\ &+\int_{P_x}(\bar{g}(x,t)-f(x,t))d\lambda(t)\\ =&\beta(x)\lambda(P_x)-c(x)(f(x,-1)-g_{-1}(x))\\ =&0.\end{aligned}$$

This has dealt with a single set $Q=Q_{nhj}$. The collection of all these sets provides a partition of $P^*\cup C_1^*$ to which Lemma 2.3 applies.

3.9. All that remains of W is the set $\mathcal{O}'\setminus P^*=\mathcal{O}''$, say. For $h,k\in\mathbb{N}$ and $r\in\mathbb{Z}$ write

$$E_{hkr}=\{(x,t)\in\mathcal{O}'': h-1\le t<h,\ \varepsilon=1/k,\ r\le 4kf(x,t)<r+1\},$$

forming an $\mathcal{A}\otimes\mathcal{B}$-measurable partition of $\mathcal{O}''$, and define

$$F_{hkr}=e(E_{hkr})=(\pi_e(E_{hkr})\times\mathbb{R})\cap E_{hkr}.$$

We have $F_{hkr}=E_{hkr}\setminus N_{hkr}$ where N_{hkr} is a fully null subset of E_{hkr}; and the sets F_{hkr},N_{hkr} $(h,k\in\mathbb{N},\ r\in\mathbb{Z})$ form an $\mathcal{A}\otimes\mathcal{B}$-measurable partition of $\mathcal{O}''$. Now Lemma 3.4 shows that each N_{hkr} satisfies the requirements of Lemma 2.3, and Lemma 3.5 shows that each F_{hkr} satisfies them too, completing the proof of Theorem 1.

3.10. COROLLARY 1: *In Theorem* 1 (2.2) *suppose further that a positive measurable function* c: $W\to(0,\infty)$ *is given. Then the theorem continues to hold if* (ii) *and* (iv) *are replaced by*

(ii)′ $|f(x,t)-c(x,t)g(x,t)|<\varepsilon(x,t)$ $((x,t)\in W)$ *and*

(iv)′ $\int_{W_x}f(x,t)d\gamma_x(t)=\int_{W_x}c(x,t)g(x,t)d\gamma_x(t)$ $(x\in X)$.

To see this, apply Theorem 1 to the functions f/c and ε/c instead of f and ε, and to the transition kernel $\bar{\gamma}$ defined by $\bar{\gamma}_x(H)=\int_{H_x}c(x,t)d\gamma_x(t)$ $(H\in\mathcal{A}\otimes\mathcal{B}|W)$ instead of γ.

COROLLARY 2: *Under the assumptions of Theorem* 1, *suppose further that all the "slice-measures" $\gamma_x(W_x)$ are finite and positive. Then there is a spectrally generating function g: $W \to \mathbb{R}$ for $\mathcal{B}(W)$ such that the function $x \mapsto \int_{W_x} g(x,t)d\gamma_x(t)$ is a spectral generator for $\mathcal{B}(X)$.*

Proof: Take a spectral generator h: $X \to \mathbb{R}$ for $\mathcal{B}(x)$, and define f: $W \to \mathbb{R}$ by $f(x,t) = h(x)/\gamma_x(W_x)$. Apply Theorem 1 to produce a spectral generator g for $\mathcal{B}(W)$, arbitrarily close to f, such that $\int_{W_x} g(x,t)d\gamma_x(t) = \int_{W_x} f(x,t)d\gamma_x(t)$. Since this last integral is $h(x)$, the result follows. ∎

As this argument shows, *if further the γ_x's are probability measures* (i.e., $\gamma_x(W_x) = 1$), *we can take g arbitrarily close to h*; for now $f(x,t) = h(x)$.

References

[1] E. van Douwen and A. H. Stone, *The topology of close approximation*, Topology and Its Applications **35** (1990), 261–275.

[2] S. Graf, *Note on Question 4*, in *Measure Theory Oberwolfach 1983*, Lecture Notes in Mathematics **1089**, Springer-Verlag, Berlin, 1984, p. 319.

[3] S. Graf and R. D. Mauldin, *A classification of disintegrations of measures*, in *Measure and Measurable Dynamics*, Contemporary Mathematics **94**, American Mathematical Society, 1989, pp. 147–158.

[4] D. Maharam and A. H. Stone, *Expressing measurable functions by one-one ones*, Advances in Mathematics **46** (1982), 151–161.

[5] D. Maharam, *A "planar" representation for generalized transition kernels*, Israel Journal of Mathematics **78** (1992), 281–307.

[6] D. Maharam, *On smoothing compact measure spaces by multiplication*, Transactions of the American Mathematical Society **204** (1975), 1–39.

[7] D. Maharam, *Fields of sets as spectral fields of functions*, in preparation.